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General relativity derived from an affine variation of a quadratic Lagrangian

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Abstract. An alternative derivation of general relativity is presented in which the field equations are derived from a gauge invariant quadratic Lagrangian by varying the affine field Γ^a_{bc} and keeping the geometry fixed. The field equations show that the Γ^a_{bc} can be considered as a Christoffel connection of a tensor g^{ab} and if, *after* the variation, the metric tensor is chosen to be this tensor, the field equations of general relativity with a cosmical constant are recovered.

In Einstein's (1915) theory of general relativity the gravitational field is identified with the geometrical tensor g_{ab} of space–time. The field equations of the theory can be derived from an action principle (Hilbert 1915)

$$\frac{\delta J}{\delta g_{ab}} = \frac{\delta}{\delta g_{ab}}\int g^{ab}R_{ab}(\sqrt{-g})\,\mathrm{d}^4x+\frac{\delta}{\delta g_{ab}}\int L_{\text{matter}}(\sqrt{-g})\,\mathrm{d}^4x = 0 \tag{1}$$

where g^{ab} is the inverse tensor of g_{ab} such that $g^{ac}g_{cb} = \delta^a_b$, and g is the determinant of the g_{ab}. An alternative derivation was provided by Palatini (1919) in which the affine field Γ^a_{bc} is defined in terms of parallel transport of a vector l^a giving a change $\mathrm{d}l^a$ with

$$\mathrm{d}l^a = -\Gamma^a_{bc}l^b\,\mathrm{d}x^c \tag{2}$$

where $\Gamma^a_{bc} = \Gamma^a_{cb}$ but is otherwise arbitrary (having forty independent components) and the Ricci tensor defined by

$$R_{ab} = \frac{\partial\Gamma^c_{ac}}{\partial x^b}-\frac{\partial\Gamma^c_{ab}}{\partial x^c}+\Gamma^c_{ad}\Gamma^d_{bc}-\Gamma^c_{ab}\Gamma^d_{cd}. \tag{3}$$

With the same action integral as (1) the theory then follows from the two sets of equations

$$\frac{\delta J}{\delta g_{ab}} = 0, \qquad \frac{\delta J}{\delta\Gamma^a_{bc}} = 0, \tag{4}$$

the second set leading to the condition that the affine derivative of $g^{ab}\sqrt{-g}$ vanishes, namely

$$\frac{\partial}{\partial x^c}(g^{ab}\sqrt{-g})+\Gamma^a_{cd}(g^{bd}\sqrt{-g})+\Gamma^b_{cd}(g^{ad}\sqrt{-g})-\Gamma^d_{cd}(g^{ab}\sqrt{-g}) = 0 \tag{5}$$

so that

$$\Gamma^a_{bc} = \frac{1}{2}g^{ad}\left(\frac{\partial g_{dc}}{\partial x^b}+\frac{\partial g_{bd}}{\partial x^c}-\frac{\partial g_{bc}}{\partial x^d}\right). \tag{6}$$

The other set then gives the standard Einstein field equations

$$R_{ab}-\tfrac{1}{2}g_{ab}R = T_{ab}. \tag{7}$$

In this formalism while the geometrical tensor g_{cb} and the affine field Γ^a_{bc} are initial independent concepts the field equations require that the theory is purely geometrodynamic, thus relating the Γ's and the g's.

This raises an interesting philosophical question that has its roots in the conventionalist philosophy of science that was so eloquently expressed by Poincaré (1902), that there is no such thing as real geometry, the only meaningful statements about the physical world are a combination of geometry and matter. The choice of geometry is arbitrary, it is just a convention and having chosen it out of the wide range of geometries known to the mathematician, a physical theory can then be expressed in this geometrical background. The physicist cannot say whether this geometry is true or false, only that on doing such and such an experiment he obtained a particular result; his experiment is not about geometry, but about the behaviour of material bodies. Of course, one may, by choosing a particular geometry, such as that found by physical bodies (or light and clocks), be able to represent the theory in a succinct form, but this is not essential to the theory. In Poincaré's words 'one geometry cannot be more true than another, it can only be more convenient'.

How is this to be reconciled with the action principle of Palatini? If we confine our attention to Riemannian geometries then there is a metric tensor η_{ab} and an affine field Γ^a_{bc}, but the η_{ab} should be an arbitrarily chosen tensor, not constrained to have zero affine derivative and therefore not related to the g_{ab} as given in equation (6). Yet since it may be more convenient to choose the η_{ab} to be so related to the Γ's, we seek a formalism that, on making this equality, reduces to general relativity. In terms of an action principle we require

$$\left(\frac{\delta J}{\delta \Gamma^a_{bc}} = 0\right)_{\eta_{ab}=g_{ab}} \tag{8}$$

to give the normal field equations of general relativity. There is an interesting point here: in this formalism the η_{ab} is set equal to the g_{ab} *after* the variation, not inside the action integral. How can this be achieved?

Now an action principle of the form (8) would give field equations

$$S^{ab}_c + S^{ba}_c = 0 \tag{9}$$

where

$$\delta J = \int S^{ab}_c(\Gamma, \partial\Gamma/\partial x, \eta, \ldots)\delta\Gamma^c_{ab}\, \mathrm{d}^4x. \tag{10}$$

As the field equations of general relativity are linear in the first derivatives of the Γ, for the field equations (9) to reduce to general relativity, S^{ab}_c must be linear in these derivatives, and so the action must be quadratic in the first derivatives of the Γ. Since the action is also an invariant we need a tensor density that is quadratic in the curvature R_{ab}, which leads us to consider

$$J = \int L_{\text{field}}(\sqrt{-\eta})\,\mathrm{d}^4x = \int \eta^{ab}\eta^{cd}(R_{ac}R_{bd}-\tfrac{1}{2}R_{ab}R_{cd})(\sqrt{-\eta})\,\mathrm{d}^4x \tag{11}$$

where η^{ab} is the inverse tensor of η_{ab} given by $\eta^{ac}\eta_{cb} = \delta^a_b$ and the R_{ab} are given in terms of the Γ's by equation (3). Such an action is invariant under the gauge transformation

$\eta_{ab} \rightarrow v(x)\eta_{ab}$. We note that the R_{ab} so defined are not necessarily symmetric, indeed

$$R_{ab} - R_{ba} = \frac{\partial \Gamma^c_{ac}}{\partial x^b} - \frac{\partial \Gamma^c_{bc}}{\partial x^a} = 2F_{ab}. \tag{12}$$

We have at this stage two possible routes to follow. We could retain the F_{ab} and hope to identify it with the electromagnetic field, since in the purely antisymmetric case the action (11) reduces to

$$J = \int F^{ab}F_{ab}(\sqrt{-\eta})\,\mathrm{d}^4x \tag{13}$$

which is just the action of the electromagnetic field, but we would expect this to lead us into the quagmire of unified field theory. Alternatively, at this stage we can insist on $F_{ab} = 0$ and explore its consequences. The first possibility is deferred to later work and we shall here take the second route. Certainly without an electromagnetic source term in our theory $F_{ab} = 0$ is a possible solution of our field equations.

The field equations are obtained by varying the action with respect to the Γ's; for our action (11) this is quite straightforward and yields (Einstein 1923)

$$H^{ab}_{:e} + H^{ba}_{:e} - \delta^a_e H^{bc}_{:c} - \delta^b_e H^{ac}_{:c} = 0 \tag{14}$$

where

$$H^{ab} = \frac{\partial L}{\partial R_{ab}} = (\eta^{ac}\eta^{bd} - \tfrac{1}{2}\eta^{ab}\eta^{cd})R_{cd}\sqrt{-\eta} \tag{15}$$

and $:e$ denotes the affine derivative defined by

$$H^{ab}_{:e} = \frac{\partial H^{ab}}{\partial x^e} + \Gamma^a_{ec}H^{cb} + \Gamma^b_{ec}H^{ac} - \Gamma^c_{ec}H^{ab}. \tag{16}$$

The forty field equations (14) determine the forty connections Γ^a_{bc} for any given metric η_{ab}. Equation (14) is symmetrized since we postulated *a priori* that the connections were symmetric, $\Gamma^a_{bc} = \Gamma^a_{cb}$, so that they are not independent quantities in the field equations obtained by setting $\delta J/\delta\Gamma^a_{bc} = 0$. In a more generalized theory without these symmetry properties the equivalent of equations (14) gives sixty equations for the sixty-four unknowns Γ^a_{bc}, since $A_b = \frac{1}{2}(\Gamma^a_{ba} - \Gamma^a_{ab})$ is not determined by the equations.

In our case we take $R_{ab} = R_{ba}$ which with a symmetric η_{ab} implies through equation (15) that H^{ab} is symmetric. Contracting equation (14) yields $H^{ae}_{:e} = 0$ and hence

$$H^{ab}_{:e} = [(\eta^{ac}\eta^{bd} - \tfrac{1}{2}\eta^{ab}\eta^{cd})R_{cd}\sqrt{\eta}]_{:e} = 0. \tag{17}$$

Since this equation shows that the affine derivative of a tensor density of weight one and rank two is identically zero it follows that we can define a symmetric tensor 'potential' g^{ab} with inverse g_{ab} such that $g^{ab}g_{bc} = \delta^a_c$, and the connections are defined in terms of the g's in exactly the same way as the Christoffel connections (cf equation (6)). We show this by taking

$$g^{ab} = \frac{\lambda h^{ab}}{(-\det h^{ab})^{1/2}(-\eta)^{1/2}} \tag{18}$$

where

$$h^{ab} = (\eta^{ac}\eta^{bd} - \tfrac{1}{2}\eta^{ab}\eta^{cd})R_{cd} \tag{19}$$

and λ is an arbitrary constant; substitution into equations (17) then gives

$$(g^{ab}(-\det g_{ab})^{1/2})_{:e} = 0 \tag{20}$$

which is just the same as equation (5) for the Palatini variation of the Einstein–Hilbert Lagrangian. These equations in turn imply $(-\det g_{ab})^{1/2}_{:e} = 0$ and hence $g^{ab}_{:e} = 0$ and then $g_{ab:e} = 0$ or

$$\frac{\partial g_{ab}}{\partial x^e} - \Gamma^c_{de} g_{cb} - \Gamma^c_{be} g_{ca} = 0 \tag{21}$$

where g_{ab} is the *inverse tensor of* g^{ab}, which is not the same as lowering the indices using the metric tensor η_{ab}. By choosing the indices (a, b, e) in this equation to be successively (d, c, b), (d, b, c) and (c, b, d), multiplying each equation by g^{ad}, adding the first two and subtracting the third, we find

$$\Gamma^a_{bc} = \frac{1}{2} g^{ad} \left(\frac{\partial g_{dc}}{\partial x^b} + \frac{\partial g_{bd}}{\partial x^c} - \frac{\partial g_{bc}}{\partial x^d} \right) \tag{22}$$

which is exactly the relationship between the Christoffel connections and the metric tensor in Riemannian geometry.

Turning now to our general programme which was to find a formalism in which the geometry tensor $\eta_{\mu\nu}$ was arbitrary and the fields Γ^a_{bc} determined in that geometry, but which reduced to general relativity with a special choice of metric tensor, this is exactly what we have achieved. Equation (17) gives the forty equations satisfied by the forty Γ^a_{bc} yet because they also imply equations (22) we can make a special choice $\eta_{ab} = g_{ab}$ in which case equation (17) becomes

$$R^{ab} - \tfrac{1}{2} g^{ab} R = \lambda g^{ab} \tag{23}$$

or Einstein's field equations with a cosmological constant. The constant λ is arbitrary since the field equations, while demonstrating that a g_{ab} exists, are invariant under the constant gauge transformation $g_{ab} \rightarrow \lambda g_{ab}$, since this leaves Γ^a_{bc} invariant.

To include the sources of the field, and to determine how matter responds to the field we need to add a matter action to our action integral (10). The standard matter action in general relativity is

$$L_{\text{matter}} \sqrt{-g} = \sum_i \int m_i \, \mathrm{d}s \, \delta^4(x-z) \tag{24}$$

where the integral is along the world line of each particle m_i, and $\delta^4(x-z)$ is the four-dimensional Dirac delta function. In our case we have an element of arc length ds, where

$$\mathrm{d}s^2 = \eta_{ab} \, \mathrm{d}x^a \, \mathrm{d}x^b \tag{25}$$

and the non-tensor fields Γ^a_{bc}, and the tensors η_{ab}, R_{ab}. Guided by our desire to reproduce general relativity by the special choice of $\eta_{ab} = g_{ab}$ we take

$$L_{\text{matter}} \sqrt{-\eta} = \kappa \sum_i \int m_i \left(R_{ab} \frac{\mathrm{d}x^a}{\mathrm{d}s} \frac{\mathrm{d}x^b}{\mathrm{d}s} \right) \mathrm{d}s \, \delta^4(x-z) \tag{26}$$

where κ is a coupling constant. We now take as our field equations

$$\frac{\delta}{\delta \Gamma^a_{bc}} \int (L_{\text{field}} + L_{\text{matter}})(\sqrt{-\eta}) \, \mathrm{d}^4 x = 0 \tag{27}$$

which on proceeding as in the previous case gives

$$[(\eta^{ac}\eta^{bd}-\tfrac{1}{2}\eta^{ab}\eta^{cd})R_{cd}\sqrt{-\eta}+\kappa T^{ab}\sqrt{-\eta}]_{:e}=0 \tag{28}$$

where

$$T^{ab}=\frac{1}{\sqrt{-\eta}}\sum_i\int m_i\frac{\mathrm{d}x^a}{\mathrm{d}s}\frac{\mathrm{d}x^b}{\mathrm{d}s}\delta^4(x-z)\,\mathrm{d}s. \tag{29}$$

Again, equations (28) demonstrate the existence of a potential tensor g_{ab} defined by

$$\lambda g^{ab}\sqrt{-g}=[(\eta^{ac}\eta^{bd}-\tfrac{1}{2}\eta^{ab}\eta^{cd})R_{cd}+\kappa T^{ab}]\sqrt{-\eta} \tag{30}$$

and again we may take the convenient, though conventional, choice of $\eta_{ab}=g_{ab}$ and write equation (30) as

$$R^{ab}-\tfrac{1}{2}g^{ab}R-\lambda g^{ab}=-\kappa T^{ab} \tag{31}$$

which is just Einstein's field equations including sources.

What about the motion of test particles? With no matter present equation (31) yields $R_{ab}=-\lambda g_{ab}$, and hence the matter Lagrangian (26) becomes

$$\kappa\lambda\sum_i\int m_i g_{ab}\frac{\mathrm{d}x^a}{\mathrm{d}s}\frac{\mathrm{d}x^b}{\mathrm{d}s}\mathrm{d}s\,\delta^4(x-z) \tag{32}$$

which reduces simply to

$$\kappa\lambda\sum_i\int m_i\,\mathrm{d}s\,\delta^4(x-z) \tag{33}$$

if we take $\eta_{ab}=g_{ab}$. Thus on varying the world line of a particle keeping the geometry and fields fixed, we obtain a geodesic of the g_{ab} Riemannian geometry, exactly as in general relativity.

I conclude, therefore, that the theory obtained by taking the symmetric affine connection as fields in the action

$$J=\int\left(\eta^{ab}\eta^{cd}(R_{ac}R_{bd}-\tfrac{1}{2}R_{ab}R_{cd})\sqrt{-\eta}+\sum_i\int m_iR_{ab}\frac{\mathrm{d}x^a}{\mathrm{d}s}\frac{\mathrm{d}x^b}{\mathrm{d}s}\delta^4(x-z)\,\mathrm{d}s\right)\mathrm{d}^4x \tag{34}$$

for arbitrary geometric field η_{ab}, yields field equations that demonstrate the existence of a tensor g_{ab}, as well as equations for the Γ^a_{bc} in terms of the sources. If we choose to set $\eta_{ab}=g_{ab}$ then the field equations are identical with general relativity. The theory presented here, therefore, *is* general relativity but expressed in such a way as to satisfy the conventionalist school of philosophy. The action is quadratic in the curvature, a fact that offers considerable hope for future work.

In the present theory we had to make the postulate that R_{ab} was symmetric; the solutions we obtained with this postulate are, of course, solutions of the theory obtained from the action (34) with general R_{ab}, and indeed of the theory obtained with general (not necessarily symmetric) Γ^a_{bc}, but I am not able to show that they are unique. I do not pursue this point here since I think it more profitable to pursue the general non-symmetric affine theory including an electromagnetic source term and I hope to return to this in a subsequent publication.

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